JOURNAL OF ALGEBRA 115, 450–465 (1988)

Some Tits Systems with Affine Weyl Groups in Chevalley Groups over Dedekind Domains

EIICHI ABE AND JUN MORITA

Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan

Communicated by Nagayoshi Iwahori

Received June 2, 1986

DEDICATED TO PROFESSOR NAGAYOSHI IWAHORI ON HIS 60TH BIRTHDAY

INTRODUCTION

Let A be a Dedekind domain and P be a nonzero prime ideal of A such that $P = (p)$ is principal and the canonical homomorphism $A^\times \to (A/P)^\times$ of multiplicative groups of units is surjective. Let R be the subring $A[\frac{1}{p}]$ of the quotient field of A. In this note, we give for a Chevalley group over R a structure of a Tits system with the affine Weyl group and give some applications of such a structure of the groups.

Our result contains as special cases the structure of Tits systems with affine Weyl groups in a Chevalley group over a field with a non-trivial discrete valuation due to N. Iwahori and H. Matsumoto [14] and in a Chevalley group over a Laurent polynomial ring due to J. Morita [17].

The Chevalley groups over a Laurent polynomial ring are regarded as groups associated with Kac–Moody Lie algebras of affine types, and our proof of the theorem is based on a technique used by D. Peterson and V. Kac [19] for groups associated with Kac–Moody Lie algebras. Since we treat the groups over Dedekind domains not necessarily as fields or local rings, there are some applications of the result. We can obtain a sufficient condition for some Dedekind domains to be universal or quasi-universal, and also we have a sort of Iwasawa decomposition of Chevalley groups over R.

The result of N. Iwahori and H. Matsumoto has been generalized by F. Bruhat and J. Tits [6] to more general reductive algebraic groups and it seems to be natural that our result could also be generalized to those groups.

Furthermore, recently some axioms and presentations characterizing Kac–Moody groups over fields have been given by J. Tits [26], which might shed new light on our approach.

0021-8693/88 $3.00

In Section 1, we introduce some definitions and properties related to Chevalley groups and Steinberg groups which are used in the following sections, and in Section 2 we prove our main theorem. Sections 3 and 4 are devoted to presenting some applications of the main theorem.

1. Chevalley Groups and Steinberg Groups

Let Φ be a reduced irreducible root system of rank l with Weyl group W and $\Pi = \{a_1, ..., a_l\}$ be a base of Φ. The positive and negative roots with respect to Π will be denoted by Φ^+ and Φ^-, respectively. For any roots $a, b \in \Phi$, we set $\langle b, a \rangle = 2(b, a)/(a, a)$, where $(\cdot, \cdot)$ is an inner product, which is invariant under W, in the real vector space generated by Φ. For each $a \in \Phi$, there is a reflection $\sigma_a \in W$ such that $\sigma_a(b) = b - \langle b, a \rangle a$. Set $S = \{\sigma_a; a \in \Pi\}$, then (W, S) is a *Coxeter system* (cf. [5]).

Let $\mathfrak{g}$ be the complex simple Lie algebra of type Φ and ρ be a faithful representation of $\mathfrak{g}$ on a finite dimensional complex vector space. Using ρ and a Chevalley basis of $\mathfrak{g}$, we may construct an affine group scheme $G_\rho(\Phi, \cdot)$ over $\mathbf{Z}$ called a *Chevalley–Demazure group scheme.* It depends up to isomorphism only on Φ and the weight lattice L_ρ of ρ. If L_ρ is the lattice of fundamental weights or the lattice of roots, we say that $G_\rho(\Phi, \cdot)$ is *simply connected* or of *adjoint type* and denote the groups by $G_{\mathrm{sc}}(\Phi, \cdot)$ or $G_{\mathrm{ad}}(\Phi, \cdot)$, respectively (cf. [1, 4, 22]).

Let R be a commutative ring with identity, then for any root $a \in \Phi$, there exists a monomorphism x_a of the additive group of R into $G_\rho(\Phi, R)$. We denote the image of x_a in $G_\rho(\Phi, R)$ by $U_a(R) = \{x_a(t) | t \in R\}$. The *elementary subgroup* $E_\rho(\Phi, R)$ of $G_\rho(\Phi, R)$ is defined to be the subgroup generated by $U_a(R)$ for all $a \in \Phi$. We denote by $U^+(R)$ (resp. $U^-(R)$) the subgroup of $E_\rho(\Phi, R)$ generated by $U_a(R)$ for all $a \in \Phi^+$ (resp. $a \in \Phi^-$). It is known that $G_{\mathrm{sc}}(\Phi, R) = E_{\mathrm{sc}}(\Phi, R)$ if R is a field, a semi-local ring or a Euclidean domain (cf. [1, 24]), and in general if rank >1, $E_\rho(\Phi, R)$ is normal in $G_\rho(\Phi, R)$ (cf. [25]). For each Φ, we define $K_1(\Phi, R) = G_{\mathrm{sc}}(\Phi, R)/E_{\mathrm{sc}}(\Phi, R)$. When we need not specify the representation ρ, we denote the groups $G_\rho(\Phi, R)$ and $E_\rho(\Phi, R)$ simply by $G(\Phi, R)$ and $E(\Phi, R)$, respectively.

The elements $x_a(t)$ of $E(\Phi, R)$ satisfy the relations

$$\text{(A)} \qquad x_a(s)\, x_a(t) = x_a(s+t)$$

$$\text{(B)} \qquad (x_a(s), x_b(t)) = \prod x_{ia+jb}\, (N_{abij} s^i t^j)$$

for all a, $b \in \Phi$ such that $a + b \neq 0$ and s, $t \in R$, and the product in (B) is taken over all roots of the form $ia + jb \in \Phi$ for positive integers i, j in some fixed order, and N_{abij} are integers depending only on the structure of the

Lie algebra $\mathfrak{g}$. For any root $a \in \Phi$ and any element u of the group $R^\times$ of the units in R, we define the elements of $E(\Phi, R)$ as

$$\text{(W)} \qquad w_a(u) = x_a(u)\, x_{-a}(-u^{-1})\, x_a(u)$$

$$\text{(H)} \qquad h_a(u) = w_a(u)\, w_a(-1).$$

Let $N(R)$ resp. $H(R)$ be the subgroup of $E(\Phi, R)$ generated by $w_a(u)$ resp. $h_a(u)$ for all $a \in \Phi$ and $u \in R^\times$. Then $H(R)$ is normal in $N(R)$ and $N(R)/H(R) \simeq W$. Put $T(R) = \mathrm{Hom}(L_\rho, R^\times)$. Then $H(R) \subset T(R) \subset G(\Phi, R)$ naturally and $T(R)$ normalizes $U^\pm(R)$ (cf. [1, 4]).

For a root system Φ of rank > 1, the *Steinberg group* $\mathrm{St}(\Phi, R)$ of type Φ over R is defined to be the group generated by symbols $\hat{x}_a(t)$ for all $a \in \Phi$ and $t \in R$ subject to the relations (A), (B), where $x_a(t)$ are replaced by $\hat{x}_a(t)$. For a root $a \in \Phi$ and an element $u \in R^\times$, we define $\hat{w}_a(u)$ and $\hat{h}_a(u)$ in $\mathrm{St}(\Phi, R)$ similarly as (W) and (H) in $E(\Phi, R)$. For a root system Φ of rank 1, $\mathrm{St}(\Phi, R)$ is defined to be the group generated by symbols $\hat{x}_a(t)$ for all $a \in \Phi$ and $t \in R$ subject to the relations (A) and

$$\text{(B}'\text{)} \qquad \hat{w}_a(u)\, \hat{x}_a(t)\, \hat{w}_a(u)^{-1} = \hat{x}_{-a}(-u^{-2}t)$$

for all $a \in \Phi$ and $u \in R^\times$, $t \in R$. Note that if rank > 1, the relation (B′) follows from (A) and (B). We have the following relations in $\mathrm{St}(\Phi, R)$.

$$\text{(R1)} \qquad \hat{w}_a(u)\, \hat{x}_b(t)\, \hat{w}_a(u)^{-1} = \hat{x}_{b'}(\eta u^{-\langle b, a\rangle} t)$$

$$\text{(R2)} \qquad \hat{w}_a(u)\, \hat{w}_b(v)\, \hat{w}_a(u)^{-1} = \hat{w}_{b'}(\eta u^{-\langle b, a\rangle} t)$$

$$\hat{w}_a(u) = \hat{w}_{-a}(-u^{-1})$$

$$\text{(R3)} \qquad \hat{w}_a(u)\, \hat{h}_b(v)\, \hat{w}_a(u)^{-1} = \hat{h}_{b'}(\eta u^{-\langle b, a\rangle} v)\, \hat{h}_{b'}(\eta u^{-\langle b, a\rangle})^{-1}$$

$$\text{(R4)} \qquad \hat{h}_a(u)\, \hat{x}_b(t)\, \hat{h}_a(u)^{-1} = \hat{x}_b(u^{\langle b, a\rangle} t)$$

$$\text{(R5)} \qquad \hat{h}_a(u)\, \hat{w}_b(v)\, \hat{h}_a(u)^{-1} = \hat{w}_b(u^{\langle b, a\rangle} v)$$

$$\hat{h}_a(u)^{-1} = \hat{h}_{-a}(u)$$

$$\text{(R6)} \qquad \hat{h}_a(u)\, \hat{h}_b(v)\, \hat{h}_a(u)^{-1} = \hat{h}_b(u^{\langle b, a\rangle} v)\, \hat{h}_b(u^{\langle b, a\rangle})^{-1}$$

for all $a, b \in \Phi$, $t \in R$, $u, v \in R^\times$, where $b' = \sigma_a(b)$, $\eta = \eta(a, b) = \pm 1$ depends only on a, b and satisfies $\eta(a, b) = \eta(a, -b)$, $\eta(a, a) = -1$. We define subgroups $\hat{N}(R)$, $\hat{H}(R)$, $\hat{U}^+(R)$, and $\hat{U}^-(R)$ of $\mathrm{St}(\Phi, R)$ similarly as those subgroups in $E(\Phi, R)$. Then, we see from the above relations that $\hat{H}(R)$ is normal in $\hat{N}(R)$, $\hat{N}(R)/\hat{H}(R) \simeq W$, and $\hat{H}(R)$ normalizes $\hat{U}^\pm(R)$.

Let $\pi\colon \mathrm{St}(\Phi, R) \to E_{\mathrm{sc}}(\Phi, R)$ be the natural homomorphism defined by $\pi(\hat{x}_a(t)) = x_a(t)$ for $a \in \Phi$ and $t \in R$, and $K_2(\Phi, R)$ be the kernel of π. Therefore we have an exact sequence

$$1 \to K_2(\Phi, R) \to \mathrm{St}(\Phi, R) \to G_{\mathrm{sc}}(\Phi, R) \to K_1(\Phi, R) \to 1.$$

In $E(\Phi, R)$, we have the relation

$$\text{(C)} \qquad h_a(u)\, h_a(v) = h_a(uv)$$

for all $a \in \Phi$ and $u, v \in R^\times$. We define an element $c_a(u, v)$ of $K_2(\Phi, R)$ by

$$c_a(u, v) = \hat{h}_a(uv)\, \hat{h}_a(u)^{-1}\, \hat{h}_a(v)^{-1}$$

and call c_a the *Steinberg symbol* for a. From (R6) we have

$$c_b(u^{\langle b, a\rangle}, v) = (\hat{h}_a(u), \hat{h}_b(v)) = c_a(v^{\langle a, b\rangle}, u)^{-1}.$$

Let $C(R)$ be the subgroup of $K_2(\Phi, R)$ generated by $c_a(u, v)$ for all $a \in \Phi$ and $u, v \in R^\times$. Then $C(R)$ is a central subgroup of $\mathrm{St}(\Phi, R)$ and generated by $c_b(u, v)$ for all $u, v \in R^\times$ with a fixed long root $b \in \Phi$, and if Φ is non-symplectic or a is short, then $c_a(u, v)$ is bimultiplicative and anti-symmetric (cf. [24] and also [22, 23]).

For any $a \in \Phi$ and $s, t \in R$ such that $1 + st \in R^\times$, in $E(\Phi, R)$, we have the relation

$$\text{(D)} \qquad x_a(s)\, x_{-a}(t) = x_{-a}(t(1+st)^{-1})\, h_a(1+st)\, x_a(s(1+st)^{-1}).$$

We define an element $d_a(s, t)$ of $K_2(\Phi, R)$ by

$$d_a(s, t) = \hat{x}_{-a}(-t(1+st)^{-1})\, \hat{x}_a(s)\, \hat{x}_{-a}(t)\, \hat{x}_a(-s(1+st)^{-1})\, \hat{h}_a(1+st)^{-1}$$

and call d_a the *Dennis–Stein symbol* for a. Let $D(R)$ be the normal subgroup of $\mathrm{St}(\Phi, R)$ generated by $d_a(s, t)$ for all $a \in \Phi$ and $s, t \in R$ such that $1 + st \in R^\times$. It can be shown that if rank $\Phi > 1$, $d_a(s, t)$ is central in $\mathrm{St}(\Phi, R)$ (cf. Appendix) and therefore $D(R)$ is a central subgroup generated by $d_a(s, t)$ for all $a \in \Phi$ and $s, t \in R$ such that $1 + st \in R^\times$. If $t = 0$, then $d_a(s, 0) = 1$ and if $t \in R^\times$, from relation (W), we have $d_a(s, t) = c_a(1 + st, t)$. Therefore, for any $u, v \in R^\times$, $c_a(u, v) = d_a((u-1)\, v^{-1}, v)$. Thus, $C(R)$ is a subgroup of $D(R)$, namely $C(R) < D(R) < K_2(\Phi, R)$ and $E_{\mathrm{sc}}(\Phi, R) \simeq \mathrm{St}(\Phi, R)/K_2(\Phi, R)$.

Next, we define $E_{\mathrm{u}}(\Phi, R) = \mathrm{St}(\Phi, R)/C(R)$, and $E_{\mathrm{q}}(\Phi, R) = \mathrm{St}(\Phi, R)/D(R)$, and we call R *universal* or *quasi-universal* for Φ if $E_{\mathrm{u}}(\Phi, R) \simeq E_{\mathrm{sc}}(\Phi, R)$ or $E_{\mathrm{q}}(\Phi, R) \simeq E_{\mathrm{sc}}(\Phi, R)$, respectively. For the groups of type A_n, universality and quasi-universality for GE_n were defined by J. Silvester [21], and universality for GE_n $(n \geqslant 2)$ and quasi-universality for GE_n $(n \geqslant 3)$ are equivalent to the definitions given here for $\Phi = A_l$ $(l \geqslant 1)$ and for $\Phi = A_l$ $(l \geqslant 2)$, respectively (cf. K. Dennis–M. Stein [8]). Note that fields, polynomial rings of one variable over fields and the ring of integers, are universal for all Φ (cf. R. Steinberg [24], U. Rehmann [20], H. Behr [2, 3], J. Hurrelbrink–U. Rehmann [13]).

Recall that (G, B, N, S) is called a *Tits system* if G is a group, B, N are subgroups of G, and S is a subset of $N/B \cap N$ satisfying the following axioms.

(T1) G is generated by B and N, and $B \cap N = H$ is a normal subgroup of N.

(T2) The set S generates the group $W = N/H$.

(T3) For all $s \in S$, and all $w \in W$, $wBs \subset BwB \cup BwsB$.

(T4) For all $s \in S$, $sBs \not\subset B$.

Note that the elements of W are the classes of N modulo H, and the notations Bw, BwB, etc., are permitted for they depend only on the classes. From the axioms it follows that the elements of S are of order 2. If (G, B, N, S) is a Tits system, then we obtain the *Bruhat decomposition* of G,

$$G = \bigsqcup_{w \in W} BwB \text{ (disjoint union)}$$

and $BwB = Bw'B$ if and only if $w = w'$.

Let F be a field and put $G = E(\Phi, F)$, $B^{\pm} = H(F)\, U^{\pm}(F)$, $N = N(F)$. Then $(G, B^{\pm}, N, S)$ is a Tits system with Weyl group W of Φ. If R is not a field, $E(\Phi, R)$ does not have such a structure of a Tits system. In the next section, we will give another type of Tits system in $E(\Phi, R)$ with affine Weyl group for some Dedekind domain R.

2. Some Tits Systems with Affine Weyl Groups

Let Φ be a reduced irreducible root system of rank l $(\geqslant 1)$, and $\Pi = \{a_1, ..., a_l\}$ be a base of Φ as in Section 1. Set $\Phi_a = \Phi \times \mathbf{Z}$. For an element $\alpha = (a, n)$ of Φ_a, denote by σ_α the permutation of Φ_a defined by

$$\sigma_\alpha(\beta) = (\sigma_a(b), m - \langle b, a \rangle n)$$

for all $\beta = (b, m) \in \Phi_a$. The *affine Weyl group* W_a associated with Φ is defined to be the subgroup of the permutation group on Φ_a generated by σ_α for all $\alpha \in \Phi_a$. We shall identify Φ with the subset $\{(a, 0) \mid a \in \Phi\}$ of Φ_a and W with the subgroup of W_a generated by σ_α for all $\alpha = (a, 0)$ with $a \in \Phi$. The set Φ_a can be regarded as the set of "real" roots in the root system of an affine Lie algebra (cf. [15]). Here, we call an element of Φ_a an *affine root* (cf. [16]). For an affine root $\alpha = (a, n)$, we denote the affine root $(a, n+m)$ simply by $\alpha + m$.

Set $\alpha_i = (-a_i, 0)$ for $i = 1, ..., l$ and $\alpha_{l+1} = (a_0, 1)$, where a_0 is the highest root of Φ, and $\Pi_a = \{\alpha_1, ..., \alpha_{l+1}\}$. Then any element of α of Φ_a can be written as a linear combination $\alpha=\sum_{i=1}^{l+1} m_i\alpha_i$ with m_i $(1\leqslant i\leqslant l+1)$ integers all non-negative or all non-positive. An affine root α is called positive (resp. negative) if $m_i\geqslant 0$ (resp. $m_i\leqslant 0$) for all i $(1\leqslant i\leqslant l+1)$. Set Φ_a^+ (resp. Φ_a^-) the set of all positive (resp. negative) roots of Φ_a. Then

$$\Phi_a^+=\{(a,n)\in\Phi_a \mid a\in\Phi^-, n\geqslant 0 \quad \text{or} \quad a\in\Phi^+, n>0\}$$

$$\Phi_a^-=\{(a,n)\in\Phi_a \mid a\in\Phi^-, n<0 \quad \text{or} \quad a\in\Phi^+, n\leqslant 0\}.$$

Further, W_a is generated by $S_a=\{\sigma_a \mid \alpha\in\Pi_a\}$ and (W_a, S_a) is a Coxeter system. For an element $w\in W_a$, if $w=s_1s_2\cdots s_n$ $(s_i\in S_a,\ 1\leqslant i\leqslant n)$ is a reduced expression of w, the number $n=l(w)$ is called the length of w, and let $N(w)$ be the number of elements of the set $\{\alpha\in\Phi_a^+ \mid w(\alpha)\in\Phi_a^-\}$. Then the following proposition which is similar to the case of finite Weyl groups is known (cf. [5, 16, 17]).

PROPOSITION 1. *Let $w\in W_a$ and $s=\sigma_\alpha\in S_a$ with $\alpha\in\Pi_a$. Then*

(1) $l(w)=N(w)$.

(2) *If $l(ws)>l(w)$, then $w(\alpha)\in\Phi_a^+$.*

Now, let A be a Dedekind domain and P be a nonzero prime ideal of A. We shall call (A, P) an *admissible pair* if it satisfies the following two conditions.

(A1) $P=(p)$ is a principal ideal.

(A2) The canonical homomorphism of multiplicative groups $A^\times\to(A/P)^\times$ is surjective.

For an admissible pair (A, P), let R be the subring $A[\frac{1}{p}]$ of the quotient field of A. Any nonzero element $t\in R$ can be written uniquely $t=up^r$ with $u\in A$ prime to p and $r\in\mathbf{Z}$. We set $v(t)=r$ and define $v(0)=\infty$. Thus, we have a map $v\colon R\to\mathbf{Z}\cup\{\infty\}$ which satisfies

(V1) $$v(s+t)\geqslant\min\{v(s), v(t)\}.$$

(V2) $$v(st)=v(s)+v(t).$$

Our main object is to discuss the structure of the elementary subgroup $E(\Phi, R)$ of the Chevalley group $G(\Phi, R)$ over R. For any affine root $\alpha=(a,n)\in\Phi_a$, we set

$$x_\alpha(u)=x_a(up^n) \qquad \text{for all} \quad u\in A,$$

$$w_\alpha(u)=x_\alpha(u)\,x_{-\alpha}(-u^{-1})\,x_\alpha(u) \qquad \text{for all} \quad u\in A^\times,$$

$$h_\alpha(u)=w_\alpha(u)\,w_\alpha(-1) \qquad \text{for all} \quad u\in A^\times,$$

and

$$U_\alpha = \{x_\alpha(u) \mid u \in A\}.$$

Then, U_α is a subgroup of $E(\Phi, R)$, being isomorphic to the additive group of A, and

$$U_\alpha \supset U_{\alpha+1} \supset U_{\alpha+2} \supset \cdots, \qquad \bigcap_{i=1}^{\infty} U_{\alpha+i} = \{1\}.$$

For any affine roots $\alpha = (a, n)$, $\beta = (b, m) \in \Phi_a$ such that $a + b \neq 0$, we have

$$(U_\alpha, U_\beta) \subset \prod U_{i\alpha + j\beta},$$

where the product is taken over all affine roots of the form $i\alpha + j\beta \in \Phi_a$ for i, j positive integers in some fixed order, and for any α, $\beta \in \Phi_a$,

$$w_\alpha(u)\, U_\beta w_\alpha(u)^{-1} \subset U_{\beta'}, \qquad \beta' = \sigma_\alpha(\beta).$$

Note that the subgroup $N(R)$ resp. $H(R)$ of $E(\Phi, R)$ is generated by $w_\alpha(u)$ resp. $h_\alpha(u)$ for all $\alpha \in \Phi_a$ and $u \in A^\times$. Further, $H(A)$ is a normal subgroup of $N(R)$ and $N(R)/H(A) \simeq W_a$. Let $U^+(A, P)$ be the subgroup of $E(\Phi, A)$ generated by U_α for all $\alpha \in \Phi_a^+$ and $B(A, P)$ the subgroup of $E(\Phi, A)$ generated by $U^+(A, P)$ and $H(A)$. Then, the following theorem holds.

THEOREM 2. *$(E(\Phi, R), B(A, P), N(R), S_a)$ is a Tits system with the affine Weyl group W_a.*

COROLLARY 3. *$(E(\Phi, A), B(A, P), N(A), S)$ is a Tits system with the Weyl group W.*

First, we give some examples of the theorem.

EXAMPLES. (1) Let F be a field endowed with a non-trivial discrete valuation v, and let A be the ring of integers and P its prime ideal. Then, since A is a local domain and P is a principal ideal, (A, P) is an admissible pair. In this case, R is the field F itself. The theorem gives the Tits system in $E(\Phi, F)$ (and also in $G(\Phi, F)$) with the affine Weyl group which was given by N. Iwahori and H. Matsumoto [14].

(2) As a special case of (1), let $A = F[[X]]$ be the formal power series ring with one variable over a field F, and P the ideal of A generated by X. Then, (A, P) is an admissible pair and R is the quotient field $F((X))$ of A. The theorem gives a Tits system in $E(\Phi, F((X)))$ (and also in

$G(\Phi, F((X)))$, which is regarded as a completed Kac–Moody group associated with the affine Lie algebra (cf. [10]).

(3) Let $A = F[X]$ be the polynomial ring with one variable over a field F, and P the ideal of A generated by X. Then, (A, P) is an admissible pair and R is the Laurent polynomial ring $F[X, X^{-1}]$ over the field F. The theorem gives the Tits system in $E(\Phi, F[X, X^{-1}])$ with the affine Weyl group which was given by J. Morita [17]. This is also a Kac–Moody group associated with the affine Lie algebra (cf. [14, 19]).

(4) Let $\mathbf{Z}$ be the ring of integers, and $P = (2)$ or (3). Then, $(\mathbf{Z}, P)$ is an admissible pair and the theorem gives a Tits system in $E(\Phi, \mathbf{Z}[\frac{1}{p}])$, where $p = 2$ or 3. Note that $P = (2)$, (3) are the only non-trivial ideals in $\mathbf{Z}$ such that $(\mathbf{Z}, P)$ is an admissible pair. More generally, let $S' = \{p_1 \ldots, p_n\}$ be a finite set of prime numbers in $\mathbf{Z}$, and let $A = \mathbf{Z}_{S'} = \mathbf{Z}[p_1^{-1}, \ldots, p_n^{-1}]$. Take a prime number $p \notin S'$. If $\pm S'$ modulo p generates the group $(\mathbf{Z}/p\mathbf{Z})^{\times}$, then (A, P) is an admissible pair with $P = (p)$. The theorem gives a Tits system in $E(\Phi, \mathbf{Z}_S)$, where $S = \{p_1, \ldots, p_n, p\}$.

Proof of Theorem 2

The proof, like those in R. Steinberg [24] and D. Peterson and V. Kac [19], proceeds in a standard way. It is enough to show the axiom (T3) of Section 1, for the proofs of the other axioms are easy. From now on, we denote the groups $B(A, P)$, $U^+(A, P)$, $H(A)$, and $N(R)$ simply by B, U^+, H, and N, respectively. For an affine root $\alpha \in \Phi_a$, let N_α be the right coset in N modulo H of σ_α.

(1) *For any affine root* $\alpha \in \Phi_a^+$,

$$U_{-\alpha} - U_{-\alpha+1} \subset U_\alpha N_\alpha U_\alpha U_{-\alpha+1}.$$

In particular, if $\alpha \in \Pi_a$, *then* $U_{-\alpha} \subset U^+ \cup U_\alpha N_\alpha U^+$.

Proof. Let $\alpha = (a, n)$ and $x_{-\alpha}(u) = x_{-a}(up^{-n}) \in U_{-\alpha}$ with $u \in A$. Assume $x_{-\alpha}(u) \notin U_{-\alpha+1}$, then $u \notin P$. From the condition (A2), u can be written as $u = u_1 + u_2 p$ with $u_1 \in A^{\times}$ and $u_2 \in A$. Therefore, $x_{-\alpha}(u) = x_{-\alpha}(u_1)$ $x_{-\alpha+1}(u_2)$. On the other hand, $x_{-\alpha}(u_1) = x_\alpha(u_1^{-1})\, w_\alpha(-u_1^{-1})\, x_\alpha(u_1^{-1})$ from the relation (B′). Thus, we see

$$x_{-\alpha}(u) = x_\alpha(u_1^{-1})\, w_\alpha(-u_1^{-1})\, x_\alpha(u_1^{-1})\, x_{-\alpha+1}(u_2).$$

The second assertion follows from the fact that $-\alpha + 1 \in \Phi_a^+$ for all $\alpha \in \Pi_a$.

(2) *For any affine root* $\alpha \in \Pi_a$, *with* $s = \sigma_\alpha$, *let*

$$U^\alpha = \langle xyx^{-1} \mid x \in U_\alpha, y \in U_\beta, \beta \in \Phi_a^+ - \{\alpha\} \rangle.$$

Then, $U^+ = U^\alpha U_\alpha = U_\alpha U^\alpha$ *and* $sU^\alpha s^{-1} = U^\alpha$.

Proof. By definition, U^α is a normal subgroup of U^+ and $U^\alpha U_\alpha$ contains generators for U^+. Thus, we have $U^+ = U^\alpha U_\alpha = U_\alpha U^\alpha$. Now, we claim that $sU^\alpha s^{-1} = U^\alpha$. Let $x_\alpha(u) \in U_\alpha$ and $x_\beta(v) \in U_\beta$, where $\alpha = (a, n) \in \Pi_a$, $\beta = (b, m) \in \Phi_a^+ - \{\alpha\}$.

Case 1. $\langle b, a\rangle \geqslant 0$. In this case, $\{ia + jb \in \Phi \mid i, j \in \mathbf{Z}_{>0}\} \subset \{a, b, a+b\}$. Therefore,

$$sx_a(up^n)\, x_b(vp^m)\, x_a(up^n)^{-1}\, s^{-1} = sx_{a+b}(cuvp^{n+m})\, x_b(vp^m)\, s^{-1}$$

for some $c \in \mathbf{Z}$. This is contained in U^α by definition.

Case 2. $\langle b, a\rangle < 0$. In this case,

$$\begin{aligned} &sx_a(up^n)\, x_b(vp^m)\, x_a(up^n)^{-1}\, s^{-1} \\ &\quad = x_{-a}(u_1 p^{-n})\, x_{b'}(v_1 p^{m - \langle b, a\rangle n})\, x_{-a}(u_1 p^{-n})^{-1} \\ &\quad = x_{-\alpha}(u_1)\, x_{\beta'}(v_1)\, x_{-\alpha}(u_1)^{-1}, \end{aligned}$$

where $b' = \sigma_a(b)$ and $\beta' = \sigma_\alpha(\beta)$, and $u_1, v_1 \in A$. From (1), one sees $x_{-\alpha}(u_1) \subset U_{-\alpha+1}$ or $U_\alpha N_\alpha U_\alpha U_{-\alpha+1}$. Since $U_{-\alpha+1} \subset U^\alpha$ and $\langle \sigma_a(b), a\rangle > 0$, the above equation can be reduced to the first case.

(3) *$sBs^{-1} \subset B \cup BsB$ for $s \in S_a$. In particular, $B \cup BsB$ is a subgroup of $E(\Phi, R)$.*

Proof. Note that $B = HU^+$ and $sHs^{-1} = H$. Let $s = \sigma_\alpha$ with $\alpha \in \Pi_\alpha$, then from (2), $sU^+s^{-1} = U_{-\alpha}U^\alpha$. Therefore, from (1),

$$sBs^{-1} = HU_{-\alpha}U^\alpha \subset HU^+ \cup HU_\alpha sU^+ = B \cup BsB.$$

(4) *For all $w \in W_a$ and $s \in S_a$,*

$$BwBsB \subset BwsB \cup BwB.$$

Proof. Let $s = \sigma_\alpha$ with $\alpha \in \Pi_a$.

Case 1. $l(ws) > l(w)$. From (1), (2) and the fact that $w(\alpha) > 0$, we obtain

$$\begin{aligned} BwBsB = BwU_\alpha U^\alpha HsB &= BwU_\alpha w^{-1}wss^{-1}U^\alpha ss^{-1}HsB \\ &\subset BU^+wsU^\alpha HB = BwsB. \end{aligned}$$

Case 2. $l(ws) < l(w)$. Set $w' = ws$. Then $w = w's$ and $l(w's) > l(w')$. Applying Case 1 with (3), we obtain

$$\begin{aligned} BwBsB &= Bw'sBsB = Bw'(B \cup BsB)\, B \\ &\subset Bw'B \cup Bw'BsB \subset Bw'B \cup Bw'sB \\ &= BwsB \cup BwB. \end{aligned}$$

Q.E.D.

Remark. Let M_a be the right coset in N modulo H represented by $\sigma_a \in W$. Then the system $(H, (U_a, M_a)_{a \in \Phi})$ in $E(\Phi, R)$ satisfies the axioms (DR1)–(DR6), except (DR4), and (V0)–(V5) of the valued root data (donnée radicielle valuée; cf. F. Bruhat and J. Tits [6, pp. 107, 117]). Instead of (DR4), the system in $E(\Phi, R)$ satisfies the weaker condition (1) rather than (RD4), which is sufficient for $E(\Phi, R)$ to have the structure of the Tits system given here.

3. Presentations of Chevalley Groups over Some Dedekind Domains

In this section, we give an application of Theorem 2 to some presentations of Chevalley groups. Let (A, P) be an admissible pair of a Dedekind domain A and a nonzero prime ideal $P = (p)$ of A, and let $R = A[\frac{1}{p}]$. As in case of $E(\Phi, R)$, for the groups $E_u(\Phi, R)$ and $E_q(\Phi, R)$, we define the subgroups $B_u(A, P)$, $H_u(A)$, $N_u(R)$ and $B_q(A, P)$, $H_q(A)$, $N_q(R)$ of $E_u(\Phi, R)$ and $E_q(\Phi, R)$, respectively. Then, $H_u(A)$ resp. $H_q(A)$ is a normal subgroup of $N_u(R)$ resp. $N_q(R)$, and further $N_u(R)/H_u(A)$ and $N_q(R)/H_q(A)$ are isomorphic to W_a. In the same manner as that of Theorem 2, we obtain the following theorem.

THEOREM 4. *Let (A, P) be an admissible pair, $P = (p)$ and $R = A[\frac{1}{p}]$. Then*

$$(E_u(\Phi, R), B_u(A, P), N_u(R), S_a)$$

and

$$(E_q(\Phi, R), B_q(A, P), N_q(R), S_a)$$

are Tits systems.

Using the Tits systems in $E(\Phi, R)$, $E_u(\Phi, R)$ and $E_q(\Phi, R)$, we can establish the following "heredity" theorem of the universality and the quasi-universality.

THEOREM 5. *Let (A, P) be an admissible pair with $P = (p)$. If A is universal or quasi-universal for Φ, then so is $R = A[\frac{1}{p}]$.*

Proof. Let A be universal. Then, we have the following commutative diagram with the canonical group homomorphism $\bar{\pi}$ of $E_u(\Phi, R)$ onto $E_{sc}(\Phi, R)$ induced by π.

$$\begin{array}{ccc} E_u(\Phi, R) & \xrightarrow{\bar{\pi}} & E_{sc}(\Phi, R) \\ \uparrow & & \uparrow \\ E_u(\Phi, A) & \longrightarrow & E_{sc}(\Phi, A) \end{array}$$

Therefore, it follows from the universality of A that the subgroup $B_u(A, P)$ of $E_u(\Phi, R)$ is isomorphic to the subgroup $B(A, P)$ of $E_{sc}(\Phi, R)$. On the other hand,

$$E_u(\Phi, R) = B_u(A, P)\, N_u(R)\, B_u(A, P),$$
$$E_{sc}(\Phi, R) = B(A, P)\, N(R)\, B(A, P)$$

by the associated Tits systems. Hence, we see that the kernel of $\bar{\pi}$ is contained in $B_u(A, P)$, and that $\bar{\pi}$ is an isomorphism. The proof in the case when A is quasi-universal is similar. Q.E.D.

COROLLARY 6. *The Laurent polynomial ring $F[X, X^{-1}]$ over a field F is universal for all Φ.*

Proof. Let $A = F[X]$, and $P = (p)$ be the ideal of A generated by X. Then (A, P) is an admissible pair and $F[X]$ is universal for all Φ (cf. U. Rehmann [20]). Therefore, $F[X, X^{-1}]$ is universal for all Φ by Theorem 4. Q.E.D.

Remark. The universality of $F[X, X^{-1}]$ for $\Phi \neq G_2$ has been given by J. Hurrelbrink [11] in a different way. This corollary is due to J. Morita [18].

COROLLARY 7. *Let $S = \{p_1, ..., p_m\}$ be a finite set of prime numbers, and $\mathbf{Z}_S = \mathbf{Z}[p_1^{-1}, ..., p_m^{-1}]$.*

(1) *If $S = \{2\}$ or $\{3\}$, then $\mathbf{Z}_S$ is universal for all Φ.*

(2) *$\mathbf{Z}_S$ is universal for all Φ if S satisfies the following condition:*

(*) *For all $i (1 \leqslant i \leqslant m)$, $\{\pm 1, p_1, ..., p_{i-1}\}$ generates* mod p_i *the multiplicative group $(\mathbf{Z}/p_i\mathbf{Z})^\times$.*

Proof. (1) follows from the fact that $\mathbf{Z}$ is universal for all Φ, and $(\mathbf{Z}, P)$ is an admissible pair, where $P = (2)$ or (3).

(2) Let $S_i = \{p_1, ..., p_i\}$ $(1 \leqslant i \leqslant m)$ and $S_0 = \{\pm 1\}$. From Theorem 4, if $\mathbf{Z}_{S_i}$ is universal for Φ, then so is $\mathbf{Z}_{S_{i+1}}$ for all i $(0 \leqslant i \leqslant m-1)$. Since $\mathbf{Z}$ is universal for Φ, inductively we have $\mathbf{Z}_S$ is universal for Φ. Q.E.D.

Remark. For $\Phi = A_l (l > 1)$, it is known that

$$(**) \qquad K_2(n, \mathbf{Z}_S) \simeq K_2(\mathbf{Z}_S) \simeq \{\pm 1\} \oplus \coprod_{p \in S} (\mathbf{Z}/p\mathbf{Z})^\times \qquad (n = l+1),$$

and $\mathbf{Z}_S$ is quasi-universal, where $S = \{p_1, ..., p_m\}$ (cf. [12]). Now, let $S = \{p\}$. Then the order of $K_2(n, \mathbf{Z}_S)$ is $2(p-1)$. On the other hand, the order of $C(\mathbf{Z}_S)$ is at most 4. Thus, if $C(\mathbf{Z}_S) = K_2(n, \mathbf{Z}_S)$, then p must be 2 or 3. Therefore, $\mathbf{Z}[\frac{1}{p}]$ is universal for A_l ($l \geqslant 1$) if and only if $p = 2, 3$. Note that in case of $l = 1$ the result comes from M. Dunwoody [9]. Our theorem can be applied for groups of each type; for example, $\mathbf{Z}_S$ with $S = \{2, 5, 97\}$ is universal for all Φ. However, the condition ($*$) is not necessary. From the structure theorem ($**$) for $K_2(n, \mathbf{Z}_S)$ with $n > 2$, we can see that $\mathbf{Z}_S$ with $S = \{5, 7\}$ is universal for A_l ($l > 1$), but $S = \{5, 7\}$ does not satisfy the condition ($*$).

EXAMPLE. Let $\mathbf{F}_3$ be the field with three elements, $A = \mathbf{F}_3[X, X^{-1}]$ the Laurent polynomial ring over $\mathbf{F}_3$, and P the ideal of A generated by $X^2 + X - 1$. Then (A, P) is an admissible pair and A is universal for all Φ. Therefore, $R = A[\frac{1}{p}]$ with $p = X^2 + X - 1$ is universal for all Φ.

4. IWASAWA DECOMPOSITIONS

In this section, we give an application of Theorem 2 to a sort of Iwasawa decomposition of Chevalley groups over Dedekind domains. Let (A, P) be an admissible pair of a Dedekind domain A and its prime ideal $P = (p)$, and let $R = A[\frac{1}{p}]$. Then, from Theorem 2, we have

$$\begin{aligned} E(\Phi, A) &= B(A, P)\, N(A)\, B(A, P), \\ E(\Phi, R) &= B(A, P)\, N(R)\, B(A, P) \\ &= B(A, P)\, H(R)\, N(A)\, B(A, P) \\ &= B(A, P)\, H(R)\, E(\Phi, A) \\ &= E(\Phi, A)\, H(R)\, E(\Phi, A). \end{aligned}$$

Note that if A is a principal ideal domain with the quotient field K, and A' is a subring of K with $A \subset A' \subset K$, then the usual *Iwasawa decomposition* (cf. [24])

$$G(\Phi, K) = U^{\pm}(K)\, T(K)\, G(\Phi, A)$$

gives the decompositions

$$\begin{aligned} G(\Phi, A') &= U^{\pm}(A')\, T(A')\, G(\Phi, A), \\ E(\Phi, A') &= U^{\pm}(A')\, H(A')(E(\Phi, A') \cap G(\Phi, A)). \end{aligned}$$

Let (A_1, p_1) and (A_2, p_2) be two admissible pairs of Dedekind domains and their prime ideals $P_1 = (p_1)$ and $P_2 = (p_2)$. We call (A_2, P_2) an *admissible extension* of (A_1, P_1) if the following conditions are satisfied:

(A3) $A_1 \subset A_2$ and $P_1 \subset P_2$;

(A4) the canonical homomorphism of multiplicative groups $A_1^\times \to (A_2/P_2)^\times$ is surjective.

For example, $(\mathbf{Z}[\sqrt{-1}], (1+\sqrt{-1}))$, $(\mathbf{Z}[\sqrt{2}], (\sqrt{2}))$, and $(\mathbf{Z}[\frac{1}{5}], (2))$ are admissible extensions of $(\mathbf{Z}, (2))$. Also, (A_P, PA_P) is an admissible extension of (A, P) for an admissible pair (A, P).

THEOREM 8. *Let (A_2, P_2) be an admissible extension of (A_1, P_1). Then*

(1) $E(\Phi, A_2) = B(A_2, P_2)\, E(\Phi, A_1)$.

(2) *If $p_1 = p_2 = p$ and $R_i = A_i[\frac{1}{p}]$ $(i = 1, 2)$, then*

$$\begin{aligned} E(\Phi, R_2) &= B(A_2, P_2)\, E(\Phi, R_1) \\ &= B(A_2, P_2)\, H(R_1)\, E(\Phi, A_1). \end{aligned}$$

Proof. We denote the subgroups $B(A_i, P_i)$ of $E(\Phi, A_i)$ by B_i $(i = 1, 2)$. We first prove that $B_2 w B_2 = B_2 w B_1$ for all $w \in W_a$ by induction on $l(w)$. Let $s = \sigma_\alpha \in S_a$ with $\alpha \in \Pi_a$, and denote the subgroups U_α and U^α in $E(\Phi, A_i)$ defined in Section 2 by $U_\alpha(A_i)$ and $U^\alpha(A_i)$, respectively. Then

$$B_2 s B_2 = B_2 s H(A_2)\, U^\alpha(A_2)\, U_\alpha(A_2) = B_2 s U_\alpha(A_2).$$

Let $x_\alpha(u) \in U_\alpha(A_2)$ and $x_\alpha(u) \notin U_{\alpha+1}(A_2)$. Then $u \in A_2$ and $u \notin P_2$. From the condition (A4), u can be written as $u = u_1 + u_2 p_2$ with $u_1 \in A_1$ and $u_2 \in A_2$. Therefore, $x_\alpha(u) = x_{\alpha+1}(u_2)\, x_\alpha(u_1) \in U^\alpha(A_2)\, U_\alpha(A_1)$ and

$$B_2 s B_2 = B_2 s U_\alpha(A_1) = B_2 s B_1.$$

Now, let $w \in W$ and $w = w's$ with $l(w') = l(w) - 1$. Then, by induction assumption,

$$\begin{aligned} B_2 w B_2 &= B_2 w' s B_2 = B_2 w' B_2 s B_1 \\ &= B_2 w' B_1 s B_1 = B_2 w' s B_1. \end{aligned}$$

Thus, we have

$$\begin{aligned} E(\Phi, A_2) &= B_2 N(A_2)\, B_2 = B_2 N(A_1)\, B_1 \\ &= B_2 E(\Phi, A_1) \end{aligned}$$

and

$$\begin{aligned} E(\Phi, R_2) &= B_2 N(R_2)\, B_2 = B_2 N(R_1)\, B_1 \\ &= B_2 E(\Phi, R_1), \\ E(\Phi, R_2) &= B_2 H(R_1)\, N(A_1)\, B_1 \\ &= B_2 H(R_1)\, E(\Phi, A_1). \end{aligned}$$

Q.E.D.

EXAMPLES.

$$E(\Phi, \mathbf{Z}[\sqrt{-1}]) = B(\mathbf{Z}[\sqrt{-1}], (1+\sqrt{-1}))\, E(\Phi, \mathbf{Z}).$$
$$E(\Phi, \mathbf{Z}[\sqrt{2}]) = B(\mathbf{Z}[\sqrt{2}], (\sqrt{2}))\, E(\Phi, \mathbf{Z}).$$
$$E(\Phi, \mathbf{Z}[\tfrac{1}{5}]) = B(\mathbf{Z}[\tfrac{1}{5}], (2))\, E(\Phi, \mathbf{Z}).$$

COROLLARY 9. *Let (A_2, P_2) be an admissible extension of (A_1, P_1). Then,*

(1) $$B(A_2, P_2) \cap E(\Phi, A_1) = B(A_1, P_1)$$

and

$$B(A_2, P_2) \backslash E(\Phi, A_2) \simeq B(A_1, P_1) \backslash E(\Phi, A_1).$$

(2) *If $p_1 = p_2$,*

$$B(A_2, P_2) \cap E(\Phi, R_1) = B(A_1, P_1)$$

and

$$B(A_2, P_2) \backslash E(\Phi, R_2) \simeq B(A_1, P_1) \backslash E(\Phi, R_1).$$

COROLLARY 10. *Let (A_2, P_2) be an admissible extension of (A_1, P_1). Suppose that (A_2, P_2) is a local ring with the quotient field $K(=R_2)$. Then,*

$$E(\Phi, A_2) = H(A_2)\, U^-(A_2)\, U^+(P_2)\, E(\Phi, A_1),$$
$$G(\Phi, A_2) = T(A_2)\, U\ \ (A_2)\, U^+(P_2)\, E(\Phi, A_1),$$
$$E(\Phi, K) = H(A_2)\, U^-(A_2)\, U^+(P_2)\, H(R_1)\, E(\Phi, A_1) \qquad \text{if} \quad p_1 = p_2,$$
$$G(\Phi, K) = T(K)\, U^-(A_2)\, U^+(P_2)\, H(R_1)\, E(\Phi, A_1) \qquad \text{if} \quad p_1 = p_2.$$

COROLLARY 11. *Let (A, P) be an admissible pair with the quotient field K and $P = (p)$. Then*

$$G(\Phi, K) = \begin{cases} T(K)\, U^-(K)\, U^+(PA_P)\, E(\Phi, A) \\ T(K)\, U^-(A_P)\, U^+(PA_P)\, E(\Phi, A[\tfrac{1}{p}]). \end{cases}$$

APPENDIX

Let Φ be a reduced irreducible root system of rank >1, and R be a commutative ring with identity. It is known that, if $\Phi = A_l (l > 1)$, then for a Dennis–Stein symbol d_a with $a \in \Phi$,

$$d_a(s, t) = \hat{x}_{-a}(-t(1+st)^{-1})\, \hat{x}_a(s)\, \hat{x}_{-a}(t)$$
$$\times \hat{x}_a(-s(1+st)^{-1})\, \hat{h}_a(1+st)^{-1},$$

where $s, t \in R$ and $1+st \in R^\times$, is central in $\mathrm{St}(\Phi, R)$ (cf. [7]), but as far as we know there are no references to the proof of this assertion available for groups of any type Φ. We will give here an elementary proof of the fact.

PROPOSITION. *For a root $a \in \Phi$, let $X_a(\Phi, R)$ be the subgroup of* $\mathrm{St}(\Phi, R)$ *generated by $x_{\pm a}(t)$ for all $t \in R$. If rank $\Phi > 1$, then $X_a(\Phi, R) \cap K_2(\Phi, R)$ is central in* $\mathrm{St}(\Phi, R)$. *In particular, $d_a(s, t)$ is central in* $\mathrm{St}(\Phi, R)$.

Proof. We may assume that $a \in \Pi$. Let $\hat{z} = \hat{x}_{\pm a}(t_1)\,\hat{x}_{\mp a}(t_2)\cdots x_{\pm a}(t_m) \in X_a(\Phi, R) \cap K_2(\Phi, R)$. We shall show that for any root $b \in \Phi$ and $u \in R$ the equality $\hat{x}_b(u)\,\hat{z}\hat{x}_b(u)^{-1} = \hat{z}$ holds.

Case 1. $b \neq \pm a$. Let $b \in \Phi^+$ (resp. Φ^-). Then by relation (B) we obtain

$$\hat{x}_b(u)\,\hat{x}_{\pm a}(t)\,\hat{x}_b(u)^{-1} = \hat{x}_{\pm a}(t) \prod \hat{x}_{\pm ia+jb}(c_{ij}t^i u^j),$$

where the product is taken over all roots $\pm ia + jb$ with positive integers i, j in some fixed order. Note that $ia + jb \in \Phi^+$ (resp. Φ^-) and $\neq \pm a$. Therefore, we have

$$\hat{x}_b(u)\,\hat{z}\hat{x}_b(u)^{-1} = \hat{z} \cdot \hat{x}, \qquad \hat{x} \in \hat{U}^+(R) \text{ resp. } \hat{U}^-(R).$$

Since $\hat{z} \in K_2(\Phi, R) = \operatorname{Ker} \pi$, we have also $\hat{x} \in \operatorname{Ker} \pi$, where π is the canonical homomorphism of $\mathrm{St}(\Phi, R)$ onto $E_{\mathrm{sc}}(\Phi, R)$. On the other hand, the restriction of π to $\hat{U}^+(R)$ resp. $\hat{U}^-(R)$ is an isomorphism. Therefore, we see $\hat{x} = 1$, namely, $\hat{x}_b(u)\,\hat{z}\hat{x}_b(u)^{-1} = \hat{z}$.

Case 2. $b = \pm a$. Since rank $\Phi > 1$, there exist roots c, $d \neq \pm a$, such that $b = \sigma_c(d)$. Then, we can express $\hat{x}_b(u)$ as

$$\hat{x}_b(u) = \hat{w}_c(1)\,\hat{x}_d(\pm u)\,\hat{w}_c(1)^{-1},$$

where it is an element of $\mathrm{St}(\Phi, R)$ which can be written as a product of elements $\hat{x}_e(v)$ with $e \neq \pm a$. Thus, we can reduce to the first case. Q.E.D.

REFERENCES

1. E. ABE, Chevalley groups over local rings, *Tôhoku Math. J. (2)* **21** (1969), 474–494.
2. H. BEHR, Eine endliche Präsentation der symplektischen Gruppe $Sp_4(\mathbf{Z})$, *Math. Z.* **141** (1975), 47–56.
3. H. BEHR, Explizite Präsentation von Chevalleygruppen über **Z**, *Math. Z.* **141** (1975), 235–241.
4. A. BOREL, Properties and linear representations of Chevalley groups, *in* "Seminar on Algebraic Groups and Related Finite Groups," pp. 1–55, Lecture Notes in Mathematics, Vol. 131, Springer-Verlag, Berlin/New York, 1970.
5. N. BOURBAKI, "Groupes et algèbres de Lie," Chaps. IV, V, VI, Hermann, Paris, 1968.

6. F. BRUHAT AND J. TITS, Groupes réductifs sur un corps local, I. Données radicielles valuées, *Inst. Haute Etudes Sci. Publ. Math.* **41** (1972), 5–252.
7. R. K. DENNIS AND M. R. STEIN, The functors K_2: A survey of computations and problems, *in* "Algebraic K-Theory II," pp. 243–280, Lecture Notes in Mathematics, Vol. 342, Springer-Verlag, Berlin/New York, 1972.
8. R. K. DENNIS AND M. R. STEIN, K_2 of radical ideals and semi-local rings revisited, *in* "Algebraic K-Theory II," pp. 281–303, Lecture Notes in Mathematics, Vol. 342, Springer-Verlag, Berlin/New York, 1972.
9. M. J. DUNWOODY, K_2 of a Euclidean ring, *J. Pure Appl. Algebra* **7** (1976), 53–58.
10. H. GARLAND, The arithmetic theory of loop groups, *Inst. Haute Etudes Sci. Publ. Math.* **52** (1980), 5–136.
11. J. HURRELBRINK, Endlich präsentierte arithmetische Gruppen und K_2 über Laurent-Plynomringen, *Math. Ann.* **225** (1977), 123–129.
12. J. HURRELBRINK, On presentations of $SL_n(\mathbf{Z}_S)$, *Comm. Algebra* **11** (9) (1983), 937–947.
13. J. HURRELBRINK AND U. REHMANN, Eine endliche Präsentation der Gruppe $G_2(\mathbf{Z})$, *Math. Z.* **141** (1975), 234–251.
14. N. IWAHORI AND H. MATSUMOTO, On some Bruhat decompositions and the structure of the Hecke rings of p-adic Chevalley groups, *Inst. Haute Etudes Sci. Publ. Math.* **25** (1965), 5–48.
15. V. G. KAC, "Infinite Dimensional Lie Algebras," Progress in Mathematics, Vol. 44, Birkhäuser, Boston, 1983.
16. I. G. MACDONALD, Affine root systems and Dedekind's η-functions, *Invent. Math.* **15** (1972), 91–143.
17. J. MORITA, Tits' systems in Chevalley groups over Laurent polynomial rings, *Tsukuba J. Math.* **3** (1979), 41–51.
18. J. MORITA, Coverings of generalized Chevalley groups associated with affine Lie algebras, *Tsukuba J. Math.* **6** (1982), 1–8.
19. D. H. PETERSON AND V. G. KAC, Infinite flag varieties and conjugacy theorems, *Proc. Nat. Acad. Sci. USA* **80** (1983), 1778–1782.
20. U. REHMANN, Präsentation von Chevalley Gruppen über $k[t]$, Preprint 1975.
21. J. R. SILVESTER, On the GL_n of a semi-local ring, *in* "Algebraic K-Theory I," pp. 244–260, Lecture Notes in Mathematics, Vol. 966 Springer-Verlag, Berlin/New York, 1982.
22. M. R. STEIN, Generators, relations and coverings of Chevalley groups over commutative rings, *Amer. J. Math.* **93** (1971), 965–1004.
23. M. R. STEIN, Surjective stability in dimension 0 for K_2 and related functors, *Trans. Amer. Math. Soc.* **178** (1973), 165–191.
24. R. STEINBERG, "Lectures on Chevalley Groups," Yale Univ. Lecture Notes, New Haven, CT, 1968.
25. G. TADDEI, Normalité des groupes élémentaire dans les groupes de Chevalley sur un anneau, *in* "Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory Part II," *Contemp. Math.* **55** (1986), 693–710.
26. J. TITS, Uniqueness and presentation of Kac–Moody groups over fields, *J. Algebra* **105** (1987), 542–573.